Univalent Material Set Theory

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TYPES 2024

Why?

Why talk about set theory in type theory?

- Set theory is mathematics too.
  - The structures of set theory might be useful.
- HoTT may give new perspectives on sets.

This talk's perspective: How to approach higher-dimensional set theory?

Formalisation: https://git.app.uib.no/hott/hott-set-theory

Outline

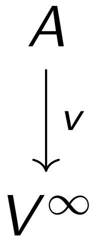
1. Models:
   - Aczel's V (aka. V^∞)
   - The iterative hierarchy (aka. V^0)
   - All the things in between (aka. V^n)
2. A couple of properties
   - Extensionality
   - Replacement

Models

V^∞

$$V^\infty := W_{A:U} A$$

- Elements on the form $\sup A\, v$ where
  - $A : U$ and
  - $v : A \to V^\infty$.
- The initial algebra for the polynomial functor $X \mapsto \sum_{A:U}(A \to X)$.
- Lives on the same type level as U (if any).



V^∞

$$V^\infty := W_{A:U} A$$

- A natural elementhood relation $x \in \sup A\, v := \sum_{a:A}(v\, a = x)$
  or: $x \in \sup A\, v := v^{-1} x$
- Used by Aczel in a setoid model of CZF.

$$\begin{array}{c} A \\ \downarrow v \\ V^\infty \end{array}$$

Sets

V^0

For V^∞, the $\in$-relation is not propositional. But we can restrict to a subtype where it is.

$$\text{is-iterative-0-type} : V^\infty \to \text{Type}$$

$$\text{is-iterative-0-type}\,(\text{sup } A\, v) :\equiv \left(\prod_{x:V^\infty} \text{is-prop}(v^{-1} x)\right) \times \left(\prod_{a:A} \text{is-iterative-0-type}\,(v\ a)\right)$$

$$V^0 :\equiv \sum_{x:V^\infty} (\text{is-iterative-0-type } x)$$

V^0 as model of set theory

$$V^0 = \sum_{x : V^\infty} (\text{is-iterative-0-type } x)$$

- V^0 is the initial algebra of the U-restricted powerset functor: $P_U^0 X = \sum_{A:U} A \hookrightarrow X$.
- V^0 is a mere set.

$$A \overset{v}{\hookrightarrow} V^0$$

V^0 as model of set theory

$$V^0 = \sum_{x : V^\infty} (\text{is-iterative-0-type } x)$$

- $\in$ naturally restricts to V^0.
- $(V^0, \in)$ models (constructive) set theory.

$$A \overset{v}{\hookrightarrow} V^0$$

Structural properties of V^0

(j.w.w. Daniel Gratzer and Anders Mörtberg)

We can also look at V^0 from a structural point of view:

- $\mathrm{El} : V^0 \to \mathrm{Type}$ defined by $\mathrm{El}(\sup A\, v) := A$
- (V^0, El) is a universe a la Tarski, closed under:
  - Σ-,Π-,Id-types
  - Inductive types such as $\mathbb{N}$ and Bool
  - Set quotients
- All decodings are definitional: $\mathrm{El}(\Pi(A, B)) \equiv \prod_{a:\mathrm{El}\ A} \mathrm{El}(B\, a)$
- Sub-universes of U generate subuniverses of V^0.

V^0

Conclusion: V^0 is a mere set universe of mere sets.

V^0

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Application: Category with Families structure on the category of sets.

See: Gratzer, Gylterud, Mörtberg , Stenholm (2024). *The Category of Iterative Sets in Homotopy Type Theory and Univalent Foundations* arXiv:2402.04893.

Higher sets

P^n

The U-restricted powerset functor P^0_U : Type → Type can be generalised as follows:

$$P^{n+1}_U : \text{Type} \to \text{Type}$$
$$P^{n+1}_U X :\equiv \sum_{A:U} A \hookrightarrow^n X$$

where $A \hookrightarrow^n X$ are the n-tructated maps into X from A.

P^n

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- $P^0_U X$ are the (U-small) subtypes X.
- $P^1_U X$ are the (U-small) covering types of X.
- $\cdots$

V^n

The initial algebra of P^n_U can be constructed just as for P^0:

$$\text{is-iter-}n{+}1\text{-type} : V^\infty \to \text{Type}$$

$$\text{is-iter-}n{+}1\text{-type}\,(\sup\ A\, v) := \left(\prod_{x:V^\infty} \text{is-n-type}\,(v^{-1}x)\right) \times \left(\prod_{a:A} \text{is-iter-}n{+}1\text{-type}\,(v\ a)\right)$$

$$V^n = \sum_{x:V^\infty} (\text{is-iter-n-type}\ x)$$

Structural properties of V^n

$$V^n = \sum_{x : V^\infty} (\text{is-iter-n-type } x)$$

- V^n is an n-type.
- $\mathsf{El} : V^n \to \mathsf{Type}$ defined by $\mathsf{El}(\sup A\, v) := A$
- (V^n, El) is a universe a la Tarski, closed under:
  - Σ-,Π-,Id-types
  - Inductive types such as $\mathbb{N}$ and Bool
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- All decodings are definitional: $\mathsf{El}(\Pi(A, B)) \equiv \prod_{a:\mathsf{El}\ A} \mathsf{El}(B\ a)$
- Sub-universes of U generate subuniverses of V^n.

Conclusion: V^n is an n-type universe of n-types.

Univalent Material Set Theory

Idea

Remember: $(V^0, \in)$ models (constructive) set theory

Question

What does $(V^n, \in)$ model?

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Answer

Univalent material set theory!

What is univalent material set theory

Univalent material set theory

- Has HoTT as its meta-theory.
- Generalises the axioms of set theory to higher type levels:
  - Level 0 is about mere sets and material sets.
  - Level 1 is about groupoids and multisets.
  - ...?
- Most axioms are indexed by type levels in range 0 to ∞.
- $x \in y$ is an $n - 1$ type, and so is $x = y$.

$\in$-structures

Definition

An $\in$-structure, $(V, \in)$ consists of

- V : Type
- $\in : V \to V \to \text{Type}$

such that the canonical map

$$x =_V y \to \prod_{z:V} (z \in x \simeq z \in y)$$

is an equivalence.

Representation and replacement

Representation of types

Definition

Given $(V, \in)$ and A : Type, a **representation** of A in $(V, \in)$ is a map $f : A \to V$.

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Definition

A an **internalisation** of f is an element $a : V$ such that for all $z : V$ we have $z \in a \simeq f^{-1}z$.

Replacement

Replacement: If a faithful representation of A in $(V, \in)$ has an in internalisation, then any faithful representation of A in $(V, \in)$ has an internalisation.

Example: Natural numbers

The von Neumann encoding gives a faithful representation $\mathbb{N} \to V$.

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The axiom of infinity says that this representation can be internalised.

With replacement, it does not matter which encoding we use.

Higher version

A representation $f : A \to V$ is $n+1$-faithful if f is n-truncted.

n-replacement: If an $n+1$ faithful representation of A can be internalised, any representation can be internalised.

Can now be applied to coverings $G \to V$.

V^1

Observation

Every n-type $A : U$ is represented in V^{n+1}.

Question

Which n-types occur in V^n?

The circle is in V^1

There is a map $f : S^1 \to V^\infty$ which maps

- base $\mapsto$ sup $\mathbb{Z}$(const $\emptyset$) and
- loop to a loop in V^1 based on succ : $\mathbb{Z} \simeq \mathbb{Z}$.

This map is 0-truncated so sup $S^1 f$ is in V^1

$$f \text{ base} = \{\cdots, \emptyset, \emptyset, \emptyset, \cdots\}$$

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- base $\mapsto \sup \mathbb{Z}(\text{const } \emptyset)$ and
- loop to a loop in V^1 based on $\text{succ} : \mathbb{Z} \simeq \mathbb{Z}$.

This map is 0-truncated so $\sup S^1\, f$ is in V^1

$$f \text{ base} = \{\cdots, \emptyset, \emptyset, \emptyset, \cdots\}$$

This argument extends to other groups.

Conclusions

- V^n is an n-type universe of n-types.
- The axioms of set theory can be extended to properties of $\in$-structures.
- For references and details:
  - https://arxiv.org/abs/2312.13024

Thank you for your attention!